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Spectral properties of two-harmonic undulator radiation

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In this paper we study the spectroscopic details of the radiation emitted by electrons propagating at relativistic velocities in undulators, whose on-axis field consists of the usual periodic term plus a contribution at a given harmonic. We present a fully analytic study and show that the undulator brightness can be written in a closed form using a new class of Bessel functions, which allows a clear understanding of the harmonic selection mechanisms pattern. We compare the analytic results with a fully numerical procedure and prove the equivalence of the two methods. We also discuss the relevance of the obtained results within the context of free-electron laser (FEL) physics.

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I. INTRODUCTION

In recent years nonconventional undulator configurations have been suggested, for various purposes, within the framework of free-electron laser (FEL) studies. Iracane and Bamas [1] have proposed the two-frequency undulator (TFU) as a device generating a laser field having both a larger extraction efficiency and a narrow spectrum. Furthermore, Schmitt and Elliott [2] conceived the two-harmonic undulator (THU) to enhance the generation of higher harmonics. More recently the authors of Ref. [3] reconsidered the THU scheme, presented a preliminary analysis of its spectral properties, and calculated the gain of a FEL operating with such an undulator device.

In this paper we develop a detailed analysis of the THU *spectroscopy* presenting an efficient algorithm of computation. The method we propose is based on the use of four-variable Bessel-type functions, which are a further generalization of a new class of Bessel functions (BF) [4] successfully exploited in previous analyses of the undulator radiation properties [5]. Since the use of these functions is crucial for the understanding of the THU spectral features, we devote the next section to a brief survey of their properties. In Sec. III we present the analytical derivation of the THU brightness. The reliability of the obtained results is checked in Sec. IV, where they are confronted with those from a fully numerical procedure. Finally, Sec. V contains concluding remarks, comparison with the TFU physics, and some considerations on the use of THU for FEL developments.

II. A BRIEF SURVEY ON THE PROPERTIES OF GENERALIZED BESSEL FUNCTIONS

Two-variable generalized BF (GBF) of the type ${}^{(m)}J_n(x,y)$ have been defined in Ref. [4] in terms of the infinite series

$$ {}^{(m)}J_n(x,y)=\sum_{l=-\infty}^{+\infty} J_{n-ml}(x)J_l(y) \; , \tag{2.1}$$

where $J_n()$ are cylinder, ordinary, first-kind BF. It can be easily proven that ${}^{(m)}J_n(x,y)$ satisfies the recurrence relations

$$\begin{aligned}
&\frac{\partial}{\partial x}\,{}^{(m)}J_n(x,y)=\tfrac{1}{2}[{}^{(m)}J_{n-1}(x,y)-{}^{(m)}J_{n+1}(x,y)] \; ,\\
&\frac{\partial}{\partial y}\,{}^{(m)}J_n(x,y)=\tfrac{1}{2}[{}^{(m)}J_{n-m}(x,y)-{}^{(m)}J_{n+m}(x,y)] \; ,\\
&2n\,{}^{(m)}J_n(x,y)=x[{}^{(m)}J_{n-1}(x,y)+{}^{(m)}J_{n+1}(x,y)]\\
&\qquad\qquad +my[{}^{(m)}J_{n-m}(x,y)+{}^{(m)}J_{n+m}(x,y)] \; ,
\end{aligned} \tag{2.2}$$

and it can be defined through the generating function

$$\sum_{n=-\infty}^{+\infty} t^n\,{}^{(m)}J_n(x,y)=\exp\left[\frac{x}{2}\left[t-\frac{1}{t}\right]+\frac{y}{2}\left[t^m-\frac{1}{t^m}\right]\right] \; , \quad |t|<\infty \; . \tag{2.3}$$

Furthermore, setting $t=e^{i\theta}$ in the above equation one obtains the generalized Jacobi-Anger expansion,

$$\sum_{n=-\infty}^{+\infty} e^{in\theta}\,{}^{(m)}J_n(x,y)=\exp\{i[x\sin\theta+y\sin(m\theta)]\} \; , \tag{2.4}$$

and thus integral representation,

$$ {}^{(m)}J_n(x,y)=\frac{1}{\pi}\int_0^{\pi} d\Phi\cos[n\Phi-x\sin\Phi-y\sin(m\Phi)] \; . \tag{2.5}$$

Finally it is worth stressing the following elementary properties:

$$\begin{aligned}
&{}^{(m)}J_n(x,0)=J_n(x) \; ,\\
&{}^{(m)}J_n(0,y)=\begin{cases} J_{n/m}(y) \; , & \text{if } \dfrac{n}{m}=\text{integer}\\ 0 \; , & \text{otherwise} \; ,\end{cases}
\end{aligned} \tag{2.6a}$$

and

$$ {}^{(m)}J_n(-x,-y)={}^{(m)}J_{-n}(x,y) \; . \tag{2.6b}$$

Together with the above GBF the further generalized forms can be introduced,

$$ {}^{(m)}_{(s,r)}J_n(x,y;u,v)=\sum_{l=-\infty}^{+\infty} {}^{(s)}J_{n-ml}(x,y){}^{(r)}J_l(u,v) \; . \tag{2.7}$$

Since, for the applications we have in mind, $s=r=2$ is enough, we will restrict the analysis to the study of the properties of the function ${}^{(m)}_{(2,2)}J_n(x,y;u,v)$ and to avoid heavy notations we will suppress the subindex (2,2). It is not difficult to realize that

(a) the *recurrence properties* of ${}^{(m)}J_n(x,y;u,v)$ are

$$\begin{aligned}
&\frac{\partial}{\partial x}\,{}^{(m)}J_n(x,y;u,v)=\tfrac{1}{2}[{}^{(m)}J_{n-1}(x,y;u,v)-{}^{(m)}J_n(x,y;u,v)] \; ,\\
&\frac{\partial}{\partial y}\,{}^{(m)}J_n(x,y;u,v)=\tfrac{1}{2}[{}^{(m)}J_{n-2}(x,y;u,v)-{}^{(m)}J_{n+2}(x,y;u,v)] \; ,\\
&\frac{\partial}{\partial u}\,{}^{(m)}J_n(x,y;u,v)=\tfrac{1}{2}[{}^{(m)}J_{n-m}(x,y;u,v)-{}^{(m)}J_{n+m}(x,y;u,v)] \; ,\\
&\frac{\partial}{\partial v}\,{}^{(m)}J_n(x,y;u,v)=\tfrac{1}{2}[{}^{(m)}J_{n-2m}(x,y;u,v)-{}^{(m)}J_{n+2m}(x,y;u,v)] \; ,\\
&2n\,{}^{(m)}J_n(x,y;u,v)=x[{}^{(m)}J_{n-1}(x,y;u,v)+{}^{(m)}J_{n+1}(x,y;u,v)]+2y[{}^{(m)}J_{n-2}(x,y;u,v)+{}^{(m)}J_{n+2}(x,y;u,v)]\\
&\qquad +m\{u[{}^{(m)}J_{n-m}(x,y;u,v)+{}^{(m)}J_{n+m}(x,y;u,v)]+2v[{}^{(m)}J_{n-2m}(x,y;u,v)+{}^{(m)}J_{n+2m}(x,y;u,v)]\} \; ;
\end{aligned}\tag{2.8}$$

(b) its *generating function* is

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} t^n\,{}^{(m)}J_n(x,y;u,v)
&=\exp\left[\frac{x}{2}\left[t-\frac{1}{t}\right]+\frac{y}{2}\left[t^2-\frac{1}{t^2}\right]+\frac{u}{2}\left[t^m-\frac{1}{t^m}\right]\right.\\
&\qquad\left.+\frac{v}{2}\left[t^{2m}-\frac{1}{t^{2m}}\right]\right] \; ;
\end{aligned}\tag{2.9}$$

(c) the corresponding *Jacobi-Anger expansion* reads

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} e^{in\theta}\,{}^{(m)}J_n(x,y;u;v)
&=\exp\{i[x\sin\theta+y\sin(2\theta)\\
&\qquad +u\sin(m\theta)+v\sin(2m\theta)]\} \; ;
\end{aligned}\tag{2.10}$$

and (d) the relevant *integral representation* is written as

$$\begin{aligned}
{}^{(m)}J_n(x,y;u,v)
&=\frac{1}{\pi}\int_0^{\pi} d\Phi\cos[n\Phi-x\sin\Phi-y\sin(2\Phi)\\
&\qquad -u\sin(m\Phi)-v\sin(2m\Phi)] \; .
\end{aligned}\tag{2.11}$$

In addition the following properties will be of particular usefulness:

$$\begin{aligned}
&{}^{(m)}J_n(x,0;u,0)={}^{(m)}J_n(x,u) \; ,\\
&{}^{(m)}J_n(0,y;0,v)=\begin{cases}{}^{(m)}J_{n/2}(y,v) \; , & n \text{ even}\\ 0, & \text{otherwise} \; ,\end{cases}
\end{aligned}\tag{2.12a}$$

and

$$ {}^{(m)}J_n(-x,-y;-u,-v)={}^{(m)}J_{-n}(x,y;u,v) \; . \tag{2.12b}$$

Numerical codes have been developed for the evaluation of the above discussed functions. These codes are based on either the series expansion or the integral representation. Both predictions are reliable and exhibit an unessential loss of precision near the zero. The precision of the codes is up to the fourteenth digit and up to the sixth digit near the zeros. In the next section we will show that the function ${}^{(m)}J_n(x,y;u,v)$ plays a central role in the study of the THU brightness.

III. ANALYSIS OF THU BRIGHTNESS

The harmonic undulator has a modified magnetic field provided by the following sum of sinusoidal forms [3]:

$$ B=\sum_{n=1}^{N} B_n\sin(k_n z) \; . \tag{3.1}$$

Albeit the problem of calculating the brightness from such a field can be solved under general conditions, here, for the sake of simplicity, assume that the field is composed by two harmonic linearly polarized fields only, namely

$$ \mathbf{B}\equiv(0,B_1\sin(k_u z)+B_h\sin(hk_u z),0) \; , \tag{3.2}$$

where h is an integer and

$$ k_u=\frac{2\pi}{\lambda_u} \; , \tag{3.3}$$

with λ_u being the undulator period.

A relativistic electron, moving in such a field, undergoes the Lorentz force and its motion will be specified by the following reduced velocities and trajectory:

$$\begin{aligned}
&\boldsymbol{\beta}\equiv(\beta_x,0,\beta_z) \; ,\\
&\beta_x\simeq-\frac{1}{\gamma}[K_1\cos(\omega_u t)+K_h\cos(h\omega_u t)] \; ,\\
&\beta_z\simeq\left\{1-\frac{1}{2\gamma^2}[1+K_1^2\cos^2(\omega_u t)+K_h^2\cos^2(h\omega_u t)\right.\\
&\qquad\left.+2K_1K_h\cos(\omega_u t)\cos(h\omega_u t)]\right\} \; ,
\end{aligned}\tag{3.4}$$

and

$$\mathbf{r}\equiv(x(t),0,z(t)) ,$$
$$x(t)=-\frac{c}{\omega_u\gamma}\left[K_1\sin(\omega_u t)+\frac{K_h}{h}\sin(h\omega_u t)\right] ,$$
$$z(t)=\beta^* ct-\frac{c}{2\gamma^2}\frac{K_1K_h}{\omega_u}\left\{\frac{1}{(1-h)}\sin[\omega_u(1-h)t]+\frac{1}{(1+h)}\sin[\omega_u(1+h)t]\right\}$$
$$-\frac{c}{8\gamma^2\omega_u}\left[K_1^2\sin(2\omega_u t)+\frac{K_h^2}{h}\sin(2h\omega_u t)\right] ,$$
$$\beta^*=1-\frac{1}{2\gamma^2}[1+\tfrac{1}{2}(K_1^2+K_h^2)] , \tag{3.5}$$

where

$$\omega_u=\frac{2\pi c}{\lambda_u} ,\quad K_1=\frac{eB_1\lambda_u}{2\pi m_0c^2} ,\quad K_h=\frac{eB_h\lambda_u}{2\pi m_0c^2h} . \tag{3.6}$$

The above equations have been derived expanding all the dynamical variables up to the order γ^{-2}.

We have all the elements to evaluate the brightness, which will be derived from the radiation integral [6]

$$\frac{d^2I}{d\omega\, d\Omega}=\frac{e^2\omega^2}{4\pi^2c}\left|\int_{-\infty}^{+\infty}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\exp\left\{i\omega\left[t-\frac{\mathbf{n}\cdot\mathbf{r}}{c}\right]\right\}dt\right|^2 , \tag{3.7}$$

where $\mathbf{n}$ is the unit observation vector (see Fig. 1) and, within the present approximation, can be written

$$\mathbf{n}\equiv(\Psi\cos\Phi,\Psi\sin\Phi,1-\tfrac{1}{2}\Psi^2) . \tag{3.8}$$

Recall that Ψ is of the order of $1/\gamma$.

Let us now proceed step by step and find a manageable expression for the various quantities appearing in the integral (3.7).

(a) *Cross products:*

$$[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})]_x\simeq\Psi\cos\Phi+\frac{1}{\gamma}[K_1\cos(\omega_u t)+K_h\cos(h\omega_u t)]$$
$$[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})]_y\simeq\Psi\sin\Phi \tag{3.9}$$
$$[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})]_z\simeq-\Psi^2-\frac{1}{\gamma}\Psi\cos\Phi[K_1\cos(\omega_u t)+K_h\cos(h\omega_u t)] ;$$

(b) *dot products:*

$$\mathbf{n}\cdot\mathbf{r}\simeq-\frac{c\Psi\cos\Phi}{\omega_u\gamma}\left[K_1\sin(\omega_u t)+\frac{K_h}{h}\sin(h\omega_u t)\right]+\beta^* ct(1-\tfrac{1}{2}\Psi^2)-\frac{c}{8\gamma^2\omega_u}\left[K_1^2\sin(2\omega_u t)+\frac{K_h^2}{h}\sin(2h\omega_u t)\right]$$
$$-\frac{c}{2\gamma^2\omega_u}K_1K_h\left[\frac{\sin[\omega_u(1-h)t]}{(1-h)}+\frac{\sin[\omega_u(1+h)t]}{(1+h)}\right] ; \tag{3.10}$$

and (c) *evaluation of the exponential:*

$$\exp\left[i\omega\left[t-\frac{\mathbf{n}\cdot\mathbf{r}}{c}\right]\right]\simeq\exp\left[i\frac{\omega t}{2\gamma^2}\left[1+\frac{K_1^2+K_h^2}{2}+\gamma^2\Psi^2\right]\right]\exp\left[i\frac{\omega\Psi\cos\Phi}{\omega_u\gamma}K_1\sin(\omega_u t)+i\frac{K_1^2\omega}{8\gamma^2\omega_u}\sin(2\omega_u t)\right.$$
$$\left.+i\frac{\omega\Psi\cos\Phi K_h}{h\omega_u\gamma}\sin(h\omega_u t)+i\frac{K_h^2\omega}{8\gamma^2h\omega_u}\sin(2h\omega_u t)\right]$$
$$\times\exp\left[i\frac{\omega}{2\gamma^2\omega_u}\frac{K_1K_h}{(1-h)}\sin[\omega_u(1-h)t]+i\frac{\omega}{2\gamma^2\omega_u}\frac{K_1K_h}{(1+h)}\sin[\omega_u(1+h)t]\right] . \tag{3.11}$$

According to the formalism developed in Sec. II we can immediately take advantage from the GBF and write

$$\exp\left[i\omega\left[t-\frac{\mathbf{n}\cdot\mathbf{r}}{c}\right]\right]\simeq\exp\left[i\frac{\omega t}{2\gamma^2}\left[1+\frac{K_1^2+K_h^2}{2}+\gamma^2\Psi^2\right]\right]$$
$$\times\sum_{n=-\infty}^{+\infty}\exp(in\omega_u t)\,{}^{(h)}J_n\left[\Psi\cos\Phi\frac{K_1}{\gamma}\frac{\omega}{\omega_u},\frac{K_1^2\omega}{8\gamma^2\omega_u};\Psi\cos\Phi\frac{K_h}{\gamma}\left[\frac{\omega}{h\omega_u}\right],\frac{K_h^2}{8\gamma^2}\left[\frac{\omega}{h\omega_u}\right]\right]$$
$$\times\exp\left[\frac{i\omega}{2\gamma^2\omega_u}\frac{K_1K_h}{(1-h)}\sin[\omega_u(1-h)t]+i\frac{\omega}{2\gamma^2\omega_u}\frac{K_1K_h}{(1+h)}\sin[\omega_u(1+h)t]\right] . \tag{3.12}$$

The problem is now that of manipulating the last exponential to get a more convenient form. Using the usual expansion formulas we find

$$\exp\left\{\frac{i\omega}{2\gamma^2\omega_u}K_1K_h\left[\frac{\sin[\omega_u(1-h)t]}{(1-h)}+\frac{\sin[\omega_u(1+h)t]}{(1+h)}\right]\right\}$$
$$=\sum_{f=-\infty}^{+\infty}\exp[if\omega_u(1-h)t]J_f\left[\frac{\omega K_1K_h}{2\gamma^2\omega_u(1-h)}\right]\sum_{g=-\infty}^{+\infty}\exp[ig\omega_u(1+h)t]J_g\left[\frac{\omega K_1K_h}{2\gamma^2\omega_u(1+h)}\right]. \quad (3.13)$$

Introducing the indices

$$s=f+g\ ,\quad \nu=g-f\ , \quad (3.14)$$

Eq. (3.13) can be rearranged as

$$\sum_{\nu=-\infty}^{+\infty}\sum_{s=-\infty}^{+\infty}e^{i(s+h\nu)\omega_u t}J_{(s-\nu)/2}\left[\frac{\omega}{2\gamma^2\omega_u}\frac{K_1K_h}{(1-h)}\right]J_{(s+\nu)/2}\left[\frac{\omega}{2\gamma^2\omega_u}\frac{K_1K_h}{(1+h)}\right], \quad (3.15)$$

thus getting, in conclusion,

$$\exp\left[\omega\left[t-\frac{\mathbf{n}\cdot\mathbf{r}}{c}\right]\right]\simeq\exp\left[\frac{i\omega t}{2\gamma^2}\left[1+\frac{K_1^2+K_h^2}{2}+\gamma^2\Psi^2\right]\right]$$
$$\times\sum_{n=-\infty}^{+\infty}e^{-in\omega_u t}\,{}^{(h)}J_n(-\zeta_\omega^{(1)},-\xi_\omega^{(1)};-\zeta_\omega^{(h)},-\xi_\omega^{(h)})$$
$$\times\sum_{s=-\infty}^{+\infty}\sum_{\nu=-\infty}^{+\infty}e^{-i(s+h\nu)\omega_u t}J_{(s-\nu)/2}(-\xi_\omega^{(-,h)})J_{(s-\nu)/2}(-\xi_\omega^{(+,h)})\ , \quad (3.16)$$

with the various arguments being specified by

$$\zeta_\omega^{(1)}=\Psi\cos\Phi\frac{K_1}{\gamma}\frac{\omega}{\omega_u}\ ,\quad \xi_\omega^{(1)}=\frac{K_1^2}{8\gamma^2}\frac{\omega}{\omega_u}\ ,\quad \zeta_\omega^{(h)}=\Psi\cos\Phi\frac{K_h}{\gamma}\left[\frac{\omega}{h\omega_u}\right],$$
$$\xi_\omega^{(h)}=\frac{K_h^2}{8\gamma^2}\left[\frac{\omega}{h\omega_u}\right],\quad \xi_\omega^{(-,h)}=\frac{K_1K_h}{(1-h)}\left[\frac{\omega}{2\gamma^2\omega_u}\right],\quad \xi_\omega^{(+,h)}=\frac{K_1K_h}{(1+h)}\left[\frac{\omega}{2\gamma^2\omega_u}\right]. \quad (3.17)$$

The problem of finding an expression for the THU brightness has been therefore solved. Performing the integration on the time we find [the superscript $(1,h)$ indicates the brightness of an undulator having the on-axis field provided by (3.2)]

$$\frac{d^2I^{(1,h)}}{d\omega\,d\Omega}=\frac{e^2}{c}N^2\left|\frac{\omega}{\omega_u}\sum_n\sum_s\sum_\nu(S_{n,s,\nu}^x,S_{n,s,\nu}^y,S_{n,s,\nu}^z)\operatorname{sinc}\left\{\pi N\left[\frac{\omega}{\omega_1}-(n+s+h\nu)\right]\right\}\exp\left\{i\pi N\left[\frac{\omega}{\omega_1}-(n+s+h\nu)\right]\right\}\right|^2, \quad (3.18)$$

where $\operatorname{sinc}(x)=(\sin x)/x$, N is the number of undulator periods,

$$\omega_1=\frac{2\gamma^2}{1+\frac{K_1^2+K_h^2}{2}+\gamma^2\Psi^2}\omega_u \quad (3.19)$$

and

$$S_{n,s,\nu}^x=\Psi(\cos\Phi)^{(h)}J_nJ_{(s-\nu)/2}J_{(s+\nu)/2}+\frac{K_1}{2\gamma}J_{(s-\nu)/2}J_{(s+\nu)/2}[{}^{(h)}J_{n-1}+{}^{(h)}J_{n+1}]$$
$$+\frac{K_h}{2\gamma}\,{}^{(h)}J_n[J_{[s-(\nu-1)]/2}J_{[s+(\nu-1)]/2}+J_{[s-(\nu+1)]/2}J_{[s+(\nu+1)]/2}]\ ,$$
$$S_{n,s,\nu}^y=\Psi(\sin\Phi)^{(h)}J_nJ_{(s-\nu)/2}J_{(s+\nu)/2}\ , \quad (3.20)$$
$$S_{n,s,\nu}^z=-\Psi^2\,{}^{(h)}J_nJ_{(s-\nu)/2}J_{(s+\nu)/2}-\frac{K_1}{2\gamma}\Psi(\cos\Phi)J_{(s-\nu)/2}J_{(s+\nu)/2}[{}^{(h)}J_{n-1}+{}^{(h)}J_{n+1}]$$
$$-\frac{K_h}{2\gamma}\Psi(\cos\Phi)^{(h)}J_n[J_{[s-(\nu-1)]/2}J_{[s+(\nu-1)]/2}+J_{[s-(\nu+1)]/2}J_{[s+(\nu+1)]/2}]\ .$$

The arguments of the BF have been omitted for conciseness.

It is worth stressing that the contribution of $(S^{z}_{n,s,\nu})^2$ can be neglected, being of the order $1/\gamma^4$. As to radiation emitted in the forward direction($\Psi=0$), the only active component is $(S^{x}_{n,s,\nu})$, which reduces to

$$\left.\frac{d^2I^{(1,h)}}{d\omega\, d\Omega}\right|_{\Psi=0}=\frac{e^2}{c}N^2\left|\frac{\omega}{\omega_u}\sum_n\sum_s\sum_\nu\left\{\frac{K_1}{2\gamma}J_{(s-\nu)/2}(-\xi_\omega^{(-,h)})J_{(s+\nu)/2}(-\xi_\omega^{(+,h)})\right.\right.$$
$$\times[{}^{(h)}J_{(n-1)/2}(-\xi_\omega^{(1)},-\xi_\omega^{(h)})+J_{(n+1)/2}(-\xi_\omega^{(1)},-\xi_\omega^{(h)})]$$
$$+\frac{K_h}{2\gamma}\,{}^{(h)}J_{n/2}(-\xi_\omega^{(1)},-\xi_\omega^{(h)})[J_{[s-(\nu-1)]/2}(-\xi_\omega^{(-,h)})$$
$$\left.\times J_{[s+(\nu-1)]/2}(-\xi_\omega^{(+,h)})+J_{[s-(\nu+1)]/2}(-\xi_\omega^{(-,h)})J_{[s+(\nu+1)]/2}(-\xi_\omega^{(+,h)})]\right\}$$
$$\left.\times\mathrm{sinc}\left\{\pi N\left[\frac{\omega}{\omega_1}-(n+s+h\nu)\right]\right\}\exp\left\{i\pi N\left[\frac{\omega}{\omega_1}-(n+s+h\nu)\right]\right\}\right|^2 . \quad (3.21)$$

Assuming that each harmonic is narrow enough that ω appearing in the argument of the GBF can be replaced with $(n+s+h\nu)\omega_1$ we have

$$\left.\frac{d^2I^{(1,h)}}{d\omega\, d\Omega}\right|_{\Psi=0}=\frac{e^2}{c}N^2\gamma^2\left[\frac{K_1}{1+\frac{K_1^2+K_h^2}{2}}\right]^2$$
$$\times\left|\sum_n\sum_s\sum_\nu(n+s+h\nu)\{J_{(s-\nu)/2}(\xi^{(-,h)})J_{(s+\nu)/2}(x^{(+,h)})[{}^{(h)}J_{(n-1)/2}(x^{(1)},\xi^{(h)})+{}^{(h)}J_{(n+1)/2}(\xi^{(1)},\xi^{(h)})]\right.$$
$$+\frac{K_h}{K_1}\,{}^{(h)}J_{n/2}(\xi^{(1)},\xi^{(h)})[J_{[s-(\nu-1)]/2}(\xi^{(-,h)})J_{[s+(\nu-1)]/2}(\xi^{(+,h)})$$
$$+J_{[s-(\nu+1)]/2}(\xi^{(-,h)})J_{[s+(\nu+1)]/2}(\xi^{(+,h)})]\}$$
$$\left.\times\mathrm{sinc}\left\{\pi N\left[\frac{\omega}{\omega_1}-(n+s+h\nu)\right]\right\}\exp\left\{i\pi N\left[\frac{\omega}{\omega_1}-(n+s+h\nu)\right]\right\}\right|^2 , \quad (3.22)$$

where

$$\xi^{(1)}=-(n+s+h\nu)\frac{K_1^2}{4}\frac{1}{1+\frac{K_1^2+K_h^2}{2}} ,$$
$$\xi^{(h)}=-(n+s+h\nu)\frac{K_h^2}{4}\frac{1}{1+\frac{K_1^2+K_h^2}{2}} , \quad (3.23)$$
$$\xi^{(\pm,h)}=-(n+s+h\nu)\frac{K_1K_h}{(1\pm h)}\frac{1}{1+\frac{K_1^2+K_h^2}{2}} .$$

It is easy to realize, just inspecting Eq. (3.22) that the various combinations of the integers $(n+s+h\nu)$ allows the emission on further peak not present in the conventional undulator brightness.

It is now worth comparing Eq. (3.22) with the on-axis radiated brightness in the linearly polarized undulator. In that case we get

$$\left.\frac{d^2I^{(1,0)}}{d\omega\, d\Omega}\right|_{\Psi=0}=\frac{e^2}{c}N^2\gamma^2\left[\frac{K}{1+\frac{K^2}{2}}\right]^2$$
$$\times\left|\sum_n n[J_{(n-1)/2}(\xi)+J_{(n+1)/2}(\xi)]\right.$$
$$\times\mathrm{sinc}\left[\pi N\left[\frac{\omega}{\omega_1}-n\right]\right]$$
$$\left.\times\exp\left[i\pi N\left[\frac{\omega}{\omega_1}-n\right]\right]\right|^2 , \quad (3.24)$$

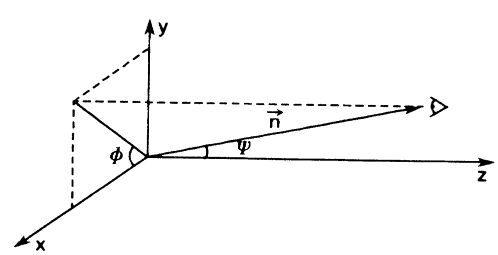

FIG. 1. Observation frame.

TABLE I. Allowed combinations of quantum numbers that correspond to the on-axis harmonic number.

			$(n+s+h\nu)$	
n	s	ν	h even	h odd
even	even	odd	even	odd
even	odd	even	odd	odd
odd	even	even	odd	odd
odd	odd	odd	even	odd

with

$$\omega_1=\frac{2\gamma^2}{1+\frac{K^2}{2}}\omega_u\ ,\quad \xi=-n\frac{K^2}{4}\frac{1}{1+\frac{K^2}{2}}\ . \tag{3.25}$$

Some analogies between the two emission process exist, but the more complicated and richer spectroscopic content of the THU case is evident. This point will be quantitatively explored in the next section.

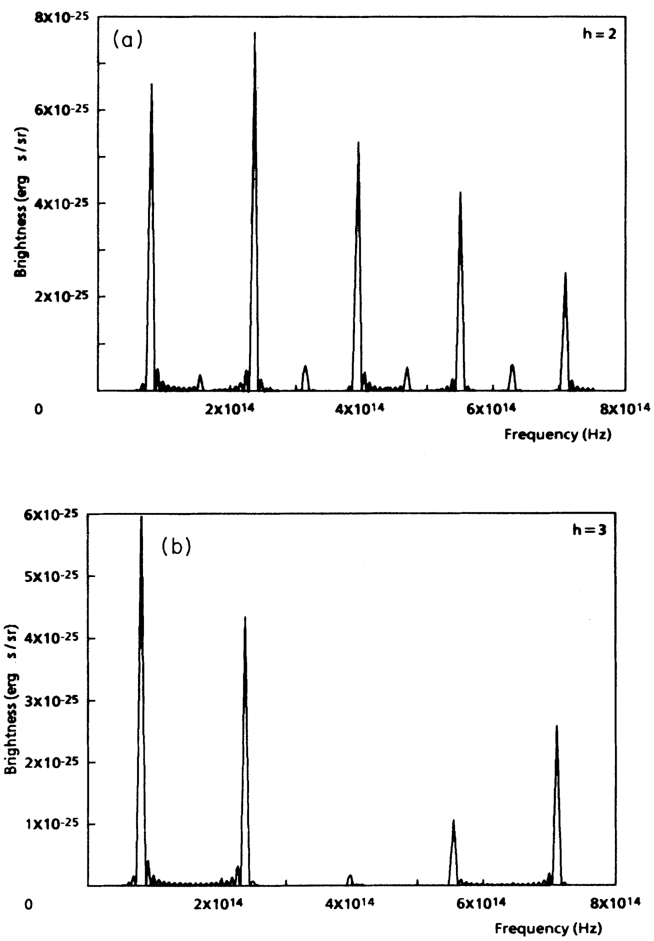

FIG. 2. (a) First nine harmonics, $h=2$, $\gamma=50$, $\lambda_u=6$ cm, $N=10$, $K_1^2+K_h^2=2$, $\psi=\Phi=0$, $K_1=1.34$; (b) First nine harmonics, $h=3$, same parameters of Fig. (a).

IV. NUMERICAL ANALYSIS

Before commenting on the results of the numerical analysis, let us explore more deeply the physics content of Eq. (3.22) and discuss what we do expect. The THU brightness will be in general characterized by three discrete numbers (n,s,ν), which specify the harmonics location.

From Eq. (3.22) and from the properties of the GBF it

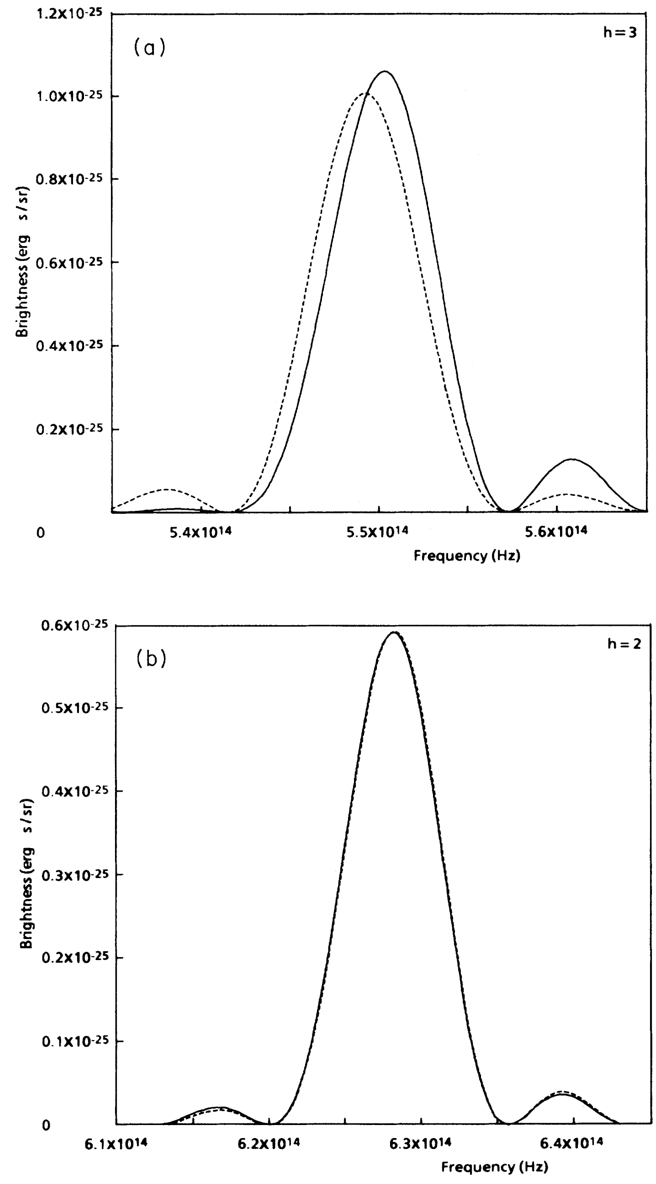

FIG. 3. (a) Seventh harmonic, $h=3$. The dashed line (— — —) denotes evaluation with GBF method $-36\le n\le 36$, $-36\le s\le 36$, $-12\le\nu\le 12$ condition $(n+s+h\nu)=7$. The solid line (——) denotes evaluation with the numerical method. (b) Eighth harmonic, $h=2$. The dashed line (— — —) denotes evaluation with GBF method $-36\le n\le 36$, $-36\le s\le 36$, $-12\le\nu\le 12$ condition $(n+s+h\nu)=8$. The solid line (——) denotes evaluation with the numerical method.

is possible to extract the combinations of quantum number which correspond to the on-axis harmonics number. The allowed combinations are given in Table I. Let us assume that the sum $(n+s+h\nu)$ describes the number of the harmonics. The first observation from the Table is that for h even the harmonics worth even number are allowed. For h odd the even number harmonics are forbidden as in the case of one frequency undulator. These conclusions are confirmed by the fully numerical calculation of the THU brightness. The on-axis emission with a frequency interval of the first eight harmonics are given in Fig. 2 for $h=2$ and 3.

The adopted numerical method has been described in Ref. [7]; it consists of a numerical integration of the electron trajectory using an initial value method and in the evaluation of the Lienard-Wiechert integral using an adoptive algorithm.

The agreement between the results obtained by the numerical method and those of the analytical computation using (3.21) in very good. For $n\in[-36,36]$, $s\in[-36,36]$, and $\nu\in[-12,12]$ the coincidence is up to the sixth digit in the large neighborhood of the peaks.

The approximation done by the assumption that the sum $(n+s+h\nu)$ corresponds to the peak number is depicted by the Fig. 3. The dotted line presents the results of the brightness calculation for the seventh and eighth harmonics region where from the sums only the combinations for which $(n+s+h\nu)$ is equal to the given harmonic number are extracted. One can see that the above approximation is good for the peak frequencies and acceptable for the nearest regions.

The obtained results may become perhaps more clear, going back to Eq. (3.15) and setting

$$s=b-h\nu\ , \tag{4.1}$$

then

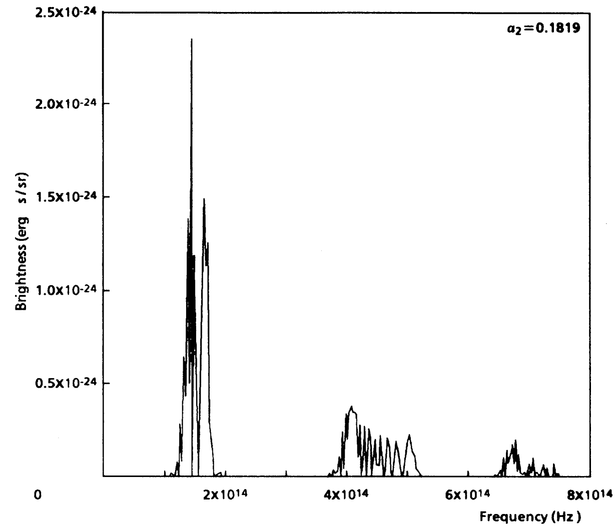

FIG. 4. Two frequency undulator spectrum. Parameters: $\lambda_u^{(1)}=5$ cm, $\lambda_u^{(2)}=5.155$ cm, $L_u=166.7$ cm, $\gamma=54.772$, $a_1=1$, $a_2=0.1819$.

$$\exp\left\{\frac{i\omega}{2\gamma^2\omega_u}K_1K_h\left[\frac{\sin[\omega_u(1-h)t]}{1-h}+\frac{\sin[\omega_u(1+h)t]}{1+h}\right]\right\}=\sum_{b=-\infty}^{+\infty}e^{ib\omega_u t}j_b\left[\frac{\omega}{2\gamma^2\omega_u}\frac{K_1K_h}{(1-h)},\frac{\omega}{2\gamma^2\omega_u}\frac{K_1K_h}{(1+h)}\right], \tag{4.2}$$

where

$$j_b(\ ,\ )=\sum_{\nu=-\infty}^{+\infty}J_{[b-(h+1)\nu]/2}\left[\frac{\omega}{2\gamma^2\omega_u}\frac{K_1K_h}{(1-h)}\right]\times J_{[b-(h-1)\nu]/2}\left[\frac{\omega}{2\gamma^2\omega_u}\frac{K_1K_h}{(1+h)}\right], \tag{4.3}$$

thus getting

$$\exp\left[i\omega\left[t-\frac{\mathbf{n}\cdot\mathbf{r}}{c}\right]\right]=\exp\left[\frac{i\omega t}{2\gamma^2}\left[1+\frac{K_1^2+K_h^2}{2}+\gamma^2\Psi^2\right]\right]\times\sum_{n=-\infty}^{+\infty}e^{-in\omega_u t}\,{}^{(h)}J_n(-\zeta_\omega^{(1)},-\xi_\omega^{(1)};-\zeta_\omega^{(h)},-\xi_\omega^{(h)})\sum_{b=-\infty}^{+\infty}e^{-ib\omega_u t}j_{-b}(\xi_\omega^{(-,h)},\xi_\omega^{(+,h)})\ . \tag{4.4}$$

Let us now introduce the GBF ${}^{(h)}R_m$ by

$$\sum_{n=-\infty}^{+\infty}e^{-in\omega_u t}\,{}^{(h)}J_n\sum_{b=-\infty}^{+\infty}e^{-ib\omega_u t}j_{-b}=\sum_{m=-\infty}^{+\infty}e^{-im\omega_u t}\,{}^{(h)}R_m(-\zeta_\omega^{(1)},-\xi_\omega^{(1)},-\zeta_\omega^{(h)},-\xi_\omega^{(h)};\xi_\omega^{(-,h)},\xi_\omega^{(+,h)})\ , \tag{4.5a}$$

where

$${}^{(h)}R_m=\sum_{b=-\infty}^{+\infty}{}^{(h)}J_{m-b}\,j_{-b}\ . \tag{4.5b}$$

The function ${}^{(h)}R_m$ can be viewed as a discrete convolution of the GBF ${}^{(h)}J_m$ on the function j_{-b}. Furthermore, if we assume that h is odd, we can set

$${}^{(2r+1)}R_m=\sum_{q=-\infty}^{+\infty}{}^{(2r+1)}J_{m-2q}\,j_{-2q}\ , \tag{4.6}$$

and write Eq. (3.22) in the simpler form,

$$\left.\frac{d^2I^{(1,2r+1)}}{d\omega\, d\Omega}\right|_{\Psi=0} = \frac{e^2}{c}N^2\gamma^2 \sum_{m=-\infty}^{+\infty} \frac{m^2K_1^2}{\left[1+\frac{K_1^2+K_h^2}{2}\right]^2} \times \{{}^{(2r+1)}R_{(m-1)/2}+{}^{(2r+1)}R_{(m+1)/2}+\frac{K_h}{K_1}({}^{(2r+1)}R_{[m-(2r+1)]/2}+{}^{(2r+1)}R_{[m+(2r+1]/2})\}^2 \times \left[\mathrm{sinc}\left[N\pi\left[\frac{\omega}{\omega_1}-m\right]\right]\right]^2 . \tag{4.7}$$

The arguments of the R functions are

$$(-m\xi^{(1)}, -m\xi^{(h)}; m\xi^{(-,h)}, m\xi^{(+,h)}) . \tag{4.8}$$

The above expression is more concise, and, perhaps, Eq. (3.22) is more physically pregnant.

V. CONCLUDING REMARKS

Before further analyzing the properties of THU radiation let us discuss their link with those of TFU brightness.

A TFU is an undulator whose on-axis field consists of two sinusoidal forms having slightly different periods. The fact that in the THU case one period is an integer multiple of the other is a significant simplification. As a consequence, at least from the mathematical point of view, the spectral properties of THU are a particular case of those of TFU.

The on-axis magnetic field of a TFU is taken of the form [1]

$$\mathbf{B} \equiv B_0[0, b(z), 0]$$

$$b(z) = a_1 \sin(k_u^{(1)}z) + a_2 \sin(k_u^{(2)}z) , \quad k_u^{(\alpha)} = \frac{2\pi}{\lambda_u^{(\alpha)}} , \quad \alpha = 1,2 \tag{5.1}$$

and the relevant on-axis brightness reads

$$\left.\frac{d^2I}{d\omega\, d\Omega}\right|_{\Psi=0} = \frac{e^2}{4\pi^2c}(\omega T)^2 \left\{ \left[\sum_l\sum_r\sum_m\sum_s B_{l,r,m,s}\frac{\sin(\Phi_{l,r,m,s}T)}{\Phi_{l,r,m,s}T}\right]^2 + \left[\sum_l\sum_r\sum_m\sum_s B_{l,r,m,s}\frac{1-\cos(\Phi_{l,r,m,s}T)}{\Phi_{l,r,m,s}T}\right]^2 \right\} , \tag{5.2}$$

where we have defined

$$\Phi_{l,r,m,s} = \frac{\omega}{2\gamma^2}\left[1+\frac{K^{*2}}{2}\right] - (l\omega_u^{(1)} + r\Delta_- + m\omega_u^{(2)} + s\Delta_+) ,$$

$$\omega^{(\alpha)} = k^{(\alpha)}c , \quad \Delta_\pm = \omega_u^{(1)} \pm \omega_u^{(2)} , \quad T = \frac{L_u}{c} , \tag{5.3}$$

$$B_{l,r,m,s} = \frac{K}{2\gamma}[a_1(J_{l+1,r,m,s}+J_{l-1,r,m,s} + \tilde{r}a_2(J_{l,r,m+1,s}+J_{l,r,m-1,s})] ,$$

$$\tilde{r} = \frac{\lambda_u^{(2)}}{\lambda_u^{(1)}} , \quad K = \frac{eB_0\lambda_u^{(1)}}{2\pi m_0c^2} , \quad K^{*2} = K^2(a_1^2+\tilde{r}^2a_2^2) .$$

The function $J_{l,r,m,s}$ is finally specified by

$$J_{l,r,m,s} = J_{l/2}(\xi_\omega^{(1)})J_r(A_\omega^{(-)})J_{m/2}(\xi_\omega^{(2)})J_s(A_\omega^{(+)}) , \tag{5.4}$$

whose arguments are

$$\xi_\omega^{(1)} = -\frac{\omega}{\omega_u^{(1)}}\frac{a_1^2K^2}{8\gamma^2} , \quad \xi_\omega^{(2)} = -\frac{\omega}{\omega_u^{(1)}}\frac{\tilde{r}^3a_2^2K^2}{8\gamma^2} ,$$

$$A_\omega^{(+)} = -\frac{\omega}{\omega_u^{(1)}}\frac{4\tilde{r}^2a_1a_2}{(1+\tilde{r})}\frac{K^2}{8\gamma^2} , \tag{5.5}$$

$$A_\omega^{(-)} = +\frac{\omega}{\omega_u^{(1)}}\frac{4\tilde{r}^2a_1a_2}{(1-\tilde{r})}\frac{K^2}{8\gamma^2} .$$

It is not difficult to realize that the indices of the summations and the BF can be rearranged to obtain the results of the previous section if $\lambda_u^{(1)}/\lambda_u^{(2)}$ is an integer. In the general TFU case the four indices l,r,m,s cannot be combined to characterize the harmonics with a single integer only. The difference between TFU and THU brightness can be understood inspecting Fig. 4. The much richer harmonic content of the former case is easily recognized; the reader interested in further details is addressed to Ref. [7].

After this digression we come back to the THU physics and discuss what we should expect if a two-harmonic device is used to generate FEL radiation. A direct application of the Madey's theorem yields an mth harmonic gain,

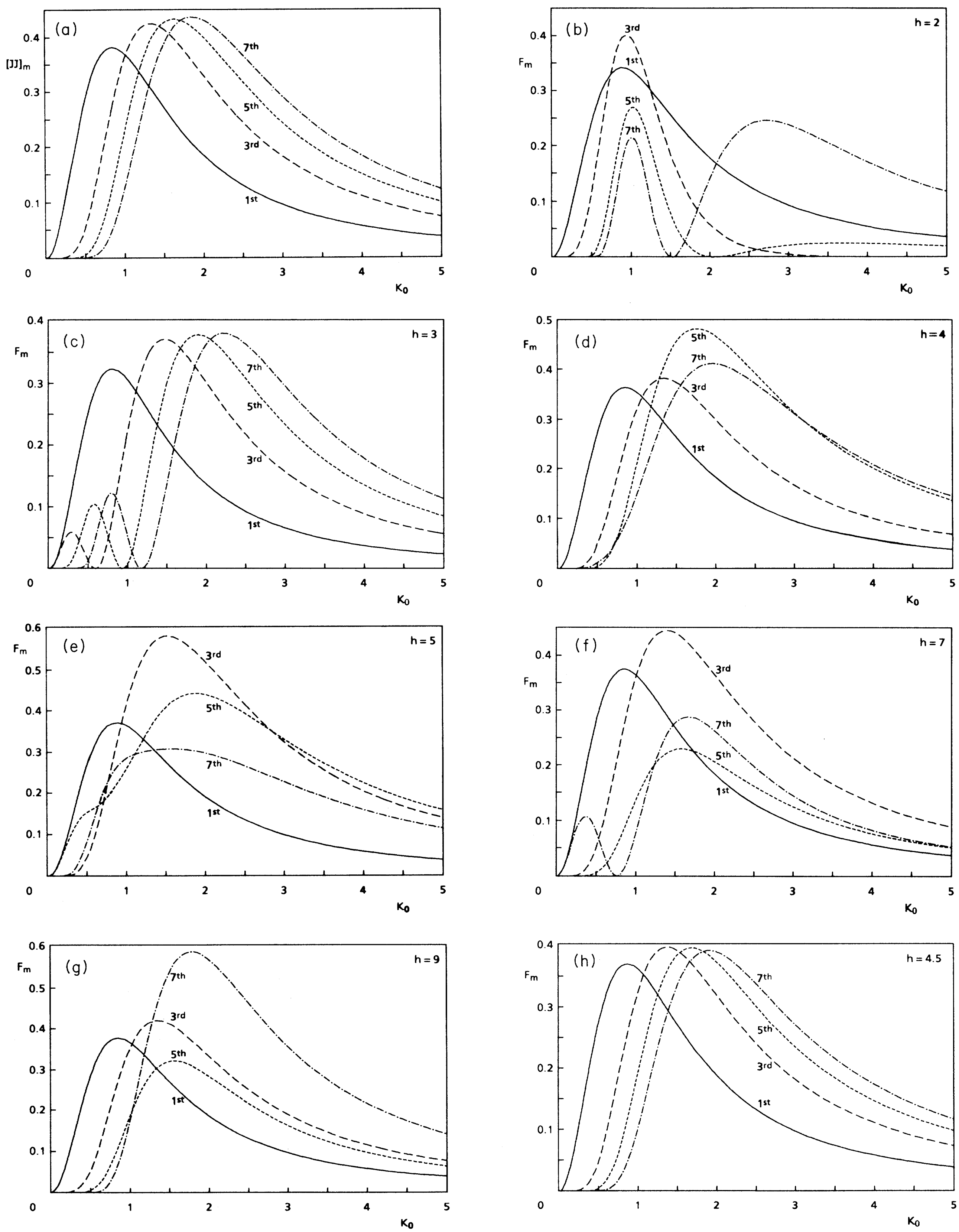

FIG. 5. (a) $[JJ]_m$ factor vs $K_0=K/\sqrt{2}$; (b) F factor vs $K_0^2=(K_1^2+K_h^2)/2$, $h=2$, $K_h=K_1/h$; (c) same as (b) $h=3$; (d) same as (b) $h=4$; (e) same as (b) $h=5$; (f) same as (b) $h=7$; (g) same as (b) $h=9$; (h) same as (b) $h=4.5$; (i) same as (b) $h=6.5$.

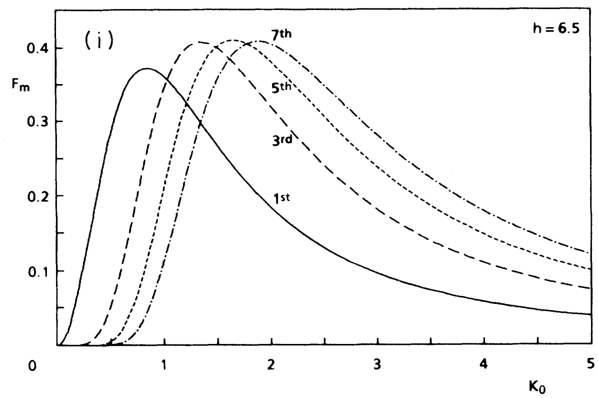

FIG. 5. (*Continued*).

$$G_m = -\pi g_m^0 \frac{\partial}{\partial \nu_m} \left[\frac{\sin \nu_m /2}{\nu_m /2} \right]^2 , \quad (5.6)$$

where $\nu_m = 2\pi N(m\omega_1 - \omega)/\omega_1$ is the detuning parameter and g_m^0 the small signal gain coefficient; namely,

$$g_m^0 = \frac{4\pi}{\gamma} \frac{\lambda_m L}{\Sigma_E} \frac{I}{I_0} F_m \left[\frac{\Delta\omega}{\omega} \right]_0^{-2}$$
$$\times F_m(\xi^{(1)}, \xi^{(h)}, \xi^{(-,h)}, \xi^{(+,h)}) , \quad (5.7)$$

with I and I_0 being the e beam and Alfvén current, respectively, F_m the filling factor, Σ_E the e-beam cross section, $(\Delta\omega/\omega)_0 = 1/2N$ the homogeneous bandwidth, and finally

$$F_m(\xi^{(1)}, \xi^{(h)}, \xi^{(-,h)}, \xi^{(+,h)})$$
$$= \xi^{(1)} m^2 \Bigg\{ {}^{(2r+1)}R_{(m-1)/2} + {}^{(2r+1)}R_{(m+1)/2} + \frac{K_h}{K_1} [{}^{(2r+1)}R_{[m-(2r+1)]/2} + {}^{(2r+1)}R_{[m+(2r+1)]/2}] \Bigg\}^2 . \quad (5.8)$$

The above function provides the coupling term to the mth harmonic and a comparison with the usual $[JJ]$ term is shown in Fig. 5.

The figures display the behavior of F_m vs $K_0 = [(K_1^2 + K_h^2)^{1/2}/2]$. The various plots are given for different values of K_h and thus for different ratios K_1/K_h, which has been assumed to be an integer in Figs. 5(a)–5(g). The figures seem to suggest that when h increases, the coupling to higher harmonics may be favored. In the case of $h = 5$ the third harmonic has a larger coupling than the first. In the case of $h = 9$ there is a substantial enhancement of the seventh harmonic. When h is not an integer [see Figs. 5(h) and 5(i)] there is no substantial difference between one and two frequency undulators.

We must stress that the present analysis does not include the homogeneous broadening induced by the betatron motion and by the emittances. This effect may substantially reduce the coupling to higher harmonics. This aspect of the problem will be discussed elsewhere.

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